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Reduced-Order Multirate Compensator Synthesis

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A method for synthesizing reduced-order multirate compensators is presented. Necessary conditions for which the compensator parameter values minimize an infinite time quadratic cost function are derived. An algorithm for finding compensator parameter values which satisfy the necessary conditions is described. This algorithm is then used to design several tip position controllers for a two-link robot arm.

Introduction

IN many cases, a multirate compensator can provide better performance than a single-rate compensator requiring the same number of real-time computations. Berg, for example, was able to reduce the steady-state rms response of states and controls to a disturbance for a simple mass-spring-mass system nearly 20% by using a multirate compensator over a single-rate compensator.[1] Numerous other examples have been provided in the literature by Berg,[1-3] Amit,[4,5] and Yang.[6] Although multirate compensators can provide improved performance over single-rate compensators, they are also, in general, more complicated to design.

The complexity of multirate compensators stems from the fact that they are by nature time varying, periodically time varying for most practical applications. Not only must designers choose multiple sampling/update rates for the compensator, but they must also determine the parameter values for a time-varying compensator.

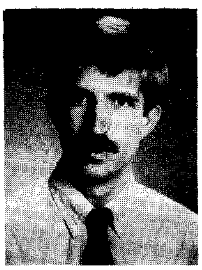
One method for designing multirate compensators is multirate linear quadratic Gaussian (LQG).[4] Multirate LQG is the multirate equivalent of single-rate LQG and is straightforward to solve because the equations governing the solution are similar to those for the single-rate case. Multirate LQG, however, results in a full-order compensator which has periodically time-varying gains. For many applications full-order, time-varying compensators are not practical.

A generalized algorithm for multirate synthesis (GAMS)[6] was developed by Yang to overcome many of the shortcomings of multirate LQG. Yang's algorithm can synthesize reduced-order multirate compensators with or without time-varying gains by using a numerical gradient-type search to find optimum compensator parameter values. His algorithm uses a finite time cost function in its problem formulation, unlike multirate and single-rate LQG which use an infinite time cost function. By using a finite time cost function, Yang's algorithm eliminates the numerical problem that arises when a destabilizing compensator is encountered during the numerical search. Even though Yang's algorithm uses a closed-form expression for the gradient, the calculations necessary to perform the gradient-type search are extremely cumbersome.

In this paper, we present a new algorithm for synthesizing reduced-order multirate compensators with or without time-varying gains. The algorithm utilizes the compensator structure of Yang's algorithm, but the problem is formulated using an infinite time, instead of a finite time, cost function. This allows us to derive necessary conditions for which the multirate compensator minimizes the cost function. The equations for the necessary conditions are fairly simple and can be solved directly using a standard nonlinear equation solver, eliminating many of the numerical complexities of Yang's algorithm.

General Multirate Compensator

Before deriving the equations governing a reduced-order multirate compensator, we will first present the structure for a general multirate compensator. We restrict our discussion for



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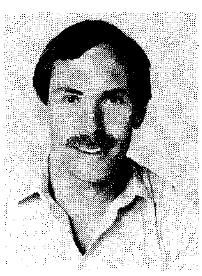

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now to compensators with time-invariant gains and sampling/update rates whose ratios are rational numbers.

A general multirate compensator is shown in Fig. 1. Each input y, output u, and state $\bar{z}$ is sampled/updated at a rate which, in general, represents the desired bandwidth of the input or output with which it is associated. The variable $\tilde{y}$ is the value of y currently available to the digital processor from the zero-order hold; while $\bar{u}$ is the current output from the digital processor which is held with a zero-order hold to form the output u. When the sampling/update rates have ratios which are rational numbers, the sampling/update schedule is periodically time varying. We define the greatest common divisor of all of the sampling/update periods as the shortest time period (*STP*) and the least common multiple of all of the sampling/update periods as the basic time period (*BTP*) (see Fig. 2).

The state equations for the multirate compensator pictured in Fig. 1 are

$$\begin{Bmatrix} \bar{z} \\ \tilde{y} \\ \tilde{u} \end{Bmatrix}_{k+1} = \begin{bmatrix} [I-s_{z,k}]+s_{z,k}\bar{A} & s_{z,k}\bar{B}[I-s_{y,k}] & 0 \\ 0 & [I-s_{y,k}] & 0 \\ s_{u,k}\bar{C} & s_{u,k}\bar{D}[I-s_{y,k}] & [I-s_{u,k}] \end{bmatrix} \times \begin{Bmatrix} \bar{z} \\ \tilde{y} \\ \tilde{u} \end{Bmatrix}_{k} + \begin{bmatrix} s_{z,k}\bar{B}s_{y,k} \\ s_{y,k} \\ s_{u,k}\bar{D}s_{y,k} \end{bmatrix} y_k \tag{1}$$

$$u_k = \begin{bmatrix} s_{u,k}\bar{C} & s_{u,k}\bar{D}[I-s_{y,k}] & [I-s_{u,k}] \end{bmatrix} \begin{Bmatrix} \bar{z} \\ \tilde{y} \\ \tilde{u} \end{Bmatrix}_{k} + [s_{u,k}\bar{D}s_{y,k}]y_k \tag{2}$$

where $\tilde{u}$ is a hold state used to model the sampler and zero-order hold between $\bar{u}$ and u. The $s_{y,k}$, $s_{z,k}$, and $s_{u,k}$ are switching matrices for y, $\bar{z}$, and u, respectively, that model the system's sampling/update activity at the start of the kth *STP*. Also, $s_{*,k}$ has the form

$$s_{*,k} = \begin{bmatrix} r_1 & 0 & 0 & \cdots & 0 \\ 0 & r_2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & r_{m_*-1} & 0 \\ 0 & \cdots & 0 & 0 & r_{m_*} \end{bmatrix}$$

where

$$r_j = \begin{cases} 1 & \text{if the } j\text{th ``*'' } (\bar{z}, y, \text{ or } u) \text{ is sampled/updated at the start of the } k\text{th } STP \\ 0 & \text{otherwise} \end{cases}$$

m_z = the number of states ($\bar{z}$)

m_y = the number of inputs (y)

m_u = the number of outputs (u)

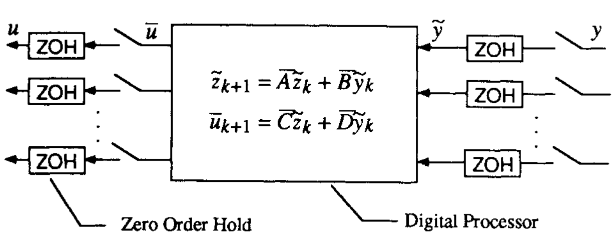


Fig. 1 A general multirate compensator.

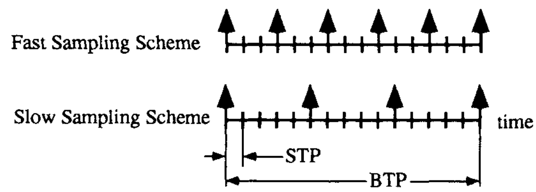


Fig. 2 Example of a multirate sampling scheme.

A more complete discussion of this compensator structure can be found in Refs. 6 and 7.

Equations (1) and (2) can be written more compactly as

$$z_{k+1} = A_k z_k + B_k y_k \tag{3}$$

$$u_k = C_k z_k + D_k y_k \tag{4}$$

where

$$z_k \equiv \begin{Bmatrix} \bar{z} \\ \tilde{y} \\ \tilde{u} \end{Bmatrix}_k$$

Equations (3) and (4) form a single-rate periodically time-varying system with a sampling rate of one *STP* and a period of one *BTP*. If $N = BTP/STP$, then $A_k = A_{k+N}$, $B_k = B_{k+N}$, $C_k = C_{k+N}$, and $D_k = D_{k+N}$.

Even though A_k, B_k, C_k, and D_k are periodically time varying, the multirate compensator gains, $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$, are time invariant. The periodicity of the multirate compensator is due to multirate sampling/updating, not the compensator gains. In the remainder of this section, we will demonstrate how the time-invariant compensator gains, $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$, can be separated from the periodic compensator matrices A_k, B_k, C_k, and D_k.

Define the composite compensator matrix as

$$P_k = \begin{bmatrix} D_k & C_k \\ B_k & A_k \end{bmatrix} \tag{5}$$

and factor P_k as follows:

$$P_k = S_{1k}\bar{P}S_{2k} + S_{3k} \tag{6}$$

where

$$\bar{P} = \begin{bmatrix} \bar{D} & \bar{C} \\ \bar{B} & \bar{A} \end{bmatrix} \tag{7}$$

$$S_{1k} = \begin{bmatrix} s_{u,k} & 0 \\ 0 & s_{z,k} \\ 0 & 0 \\ s_{u,k} & 0 \end{bmatrix} \tag{8}$$

$$S_{2k} = \begin{bmatrix} s_{y,k} & 0 & I-s_{y,k} & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \tag{9}$$

$$S_{3k} = \begin{bmatrix} 0 & 0 & 0 & I-s_{u,k} \\ 0 & I-s_{z,k} & 0 & 0 \\ s_{y,k} & 0 & I-s_{y,k} & 0 \\ 0 & 0 & 0 & I-s_{u,k} \end{bmatrix} \tag{10}$$

Equation (6) is a key result. It allows us to factor the time-invariant compensator gains, the unknown parameters we will solve for in the next section, out of the time-varying compensator.

It is important to note the difference between P_k and $\tilde{P}$ in Eq. (6). P_k (with a subscript) is a periodically time-varying matrix defined by Eq. (5). It includes all of the information about the compensator gains and the sampling/update schedule. $\tilde{P}$ is a constant matrix which contains only the gains for the compensator. P_k can be written in terms of $\tilde{P}$ and S_{1k}, S_{2k}, and S_{3k} using Eq. (6). S_{1k}, S_{2k}, and S_{3k} are periodically time-varying matrices which contain a description of the sampling/update scheme.

Derivation of the Necessary Conditions

In this section, we will use the results of the previous section to derive the necessary conditions for the reduced-order multirate compensator. The multirate problem to be solved is as follows.

Given:
the discretized plant model

$$\hat{x}_{k+1} = \hat{F}\hat{x}_k + \hat{G}\hat{u}_k + \hat{W}w_k \tag{11}$$

$$\hat{y}_k = \hat{H}\hat{x}_k + v_k \tag{12}$$

where $\hat{F}$, $\hat{G}$, $\hat{W}$, and $\hat{H}$ are obtained by discretizing the analog plant matrices at one STP; w_k and v_k are discrete-time Gaussian white noise inputs; $\hat{u}$ is the control input from the compensator; and $\hat{y}$ is the sampled sensor output.

Find:
the multirate control law with a prescribed dynamic order and sampling schedule, of the form of Eqs. (1) and (2) which minimizes a quadratic cost function of the form

$$J = \lim_{t\to\infty} \sum_{k=1}^{N} E\left\{\begin{Bmatrix} x_k \\ u_k \end{Bmatrix}^T \begin{bmatrix} Q_1 & M \\ M^T & Q_2 \end{bmatrix} \begin{Bmatrix} x_k \\ u_k \end{Bmatrix}\right\} \tag{13}$$

where E is the expected value operator, and the summation from 1 to N accounts for the fact that the closed-loop system is periodically time varying. A prescribed sampling schedule implies that the values of $s_{z,k}$, $s_{y,k}$ and $s_{u,k}$ are known.

Using Eqs. (3) and (4), it is easy to see that this problem is essentially a time-varying feedback problem—a time-invariant plant with a periodically time-varying compensator. One thing that makes this problem difficult is that the compensator has an explicit form, that of Eqs. (1) and (2), in which only certain parameters, $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$, can be adjusted to minimize J.

To solve the multirate control problem, we cast it into output feedback form and follow a derivation similar to Mukhopadhyay's for the single rate case.[8,9] Using Eqs. (3) and (4) and Eqs. (11) and (12), we write the output feedback equations

$$\begin{Bmatrix} \hat{x}_{k+1} \\ z_{k+1} \end{Bmatrix} = \begin{bmatrix} \hat{F} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{x}_k \\ z_k \end{Bmatrix} + \begin{bmatrix} \hat{G} & 0 \\ 0 & I \end{bmatrix} \begin{Bmatrix} \hat{u}_k \\ z_{k+1} \end{Bmatrix} + \begin{bmatrix} \hat{W} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} w_k \\ v_k \end{Bmatrix} \tag{14}$$

$$\begin{Bmatrix} \hat{y}_k \\ z_k \end{Bmatrix} = \begin{bmatrix} \hat{H} & 0 \\ 0 & I \end{bmatrix} \begin{Bmatrix} \hat{x}_k \\ z_k \end{Bmatrix} + \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} w_k \\ v_k \end{Bmatrix} \tag{15}$$

$$\begin{Bmatrix} \hat{u}_k \\ z_{k+1} \end{Bmatrix} = \begin{bmatrix} D_k & C_k \\ B_k & A_k \end{bmatrix} \begin{Bmatrix} \hat{y}_k \\ z_k \end{Bmatrix} \tag{16}$$

Equations (14–16) can be written more compactly as

$$x_{k+1} = Fx_k + Gu_k + W\eta_k \tag{17}$$

$$y_k = Hx_k + V\eta_k \tag{18}$$

$$u_k = P_k y_k \tag{19}$$

It is important to keep in mind that P_k in Eq. (19) corresponds to the P_k in Eq. (5), a periodically time-varying matrix which contains all of the information about the multirate compensator gains and sampling/update rates.

The closed-loop system is

$$x_{k+1} = F_{ck}x_k + G_{ck}\eta_k \tag{20}$$

where

$$F_{ck} = F + GP_kH \tag{21}$$

$$G_{ck} = W + GP_kH \tag{22}$$

The state covariance propagation for this system obeys

$$X_{k+1} = F_{ck}X_kF_{ck}^T + G_{ck}RG_{ck}^T \tag{23}$$

where

$$X_k = E\{x_kx_k^T\}, \qquad R = E\{\eta_k\eta_k^T\}$$

Equations (20–22) represent a periodically time-varying system with a period of one BTP. We can generate a single-rate system by repeated application of Eq. (20) over one BTP.[10] The single-rate system can be written as

$$x_{k+N} = F_{bk}x_k + G_{bk}\eta_{bk} \tag{24}$$

where

$$F_{bk} = F_{c(k+N-1)}F_{c(k+N-2)}F_{c(k+N-3)}\cdots F_{ck} \tag{25}$$

$$G_{bk} = \left[F_{c(k+N-1)}F_{c(k+N-2)}\cdots F_{c(k+1)}G_{ck} \mid F_{c(k+N-1)}F_{c(k+N-2)}\cdots F_{c(k+2)}G_{c(k+1)} \mid \cdots \mid G_{c(k+N-1)}\right] \tag{26}$$

$$\eta_{bk} = \begin{bmatrix} \eta_k \\ \eta_{k+1} \\ \vdots \\ \eta_{k+N-1} \end{bmatrix}$$

This single-rate system has exactly the same values for x as the periodically time-varying, closed-loop system at each BTP. However, the values of x at the intermediate STP are lost because x is incremented by N in Eq. (24) but only by 1 in Eq. (20). There are N such single-rate systems associated with Eq. (20). They can be written as

$$x_{k+N+i} = F_{b(k+i)}x_{k+i} + G_{b(k+i)}\eta_{k+i}, \quad \text{for } i = 1,2,\ldots,N \tag{27}$$

If F_{bk} is stable, then the periodically time-varying system Eq. (20) is stable.[11] We can calculate the steady-state covariance for x using the following Lyapunov equations:

$$X_k = F_{bk}X_kF_{bk}^T + G_{bk}R_bG_{bk}^T, \qquad \text{for } k = 1,2,\ldots,N \tag{28}$$

$$R_b = \begin{bmatrix} R & 0 & \cdots & 0 \\ 0 & R & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & R \end{bmatrix}$$

Note that X_k is periodic, that is it varies within one BTP, but from BTP to BTP $X_k = X_{k+N}$. Once we have calculated X_k at any k using Eq. (28), we can use Eq. (23) to propagate it over the BTP. This eliminates the need to solve Eq. (28) N times.

Now, using Eqs. (23) and (13), and the properties of the trace (Tr) operator, we can write the cost function for the stabilized system as (see Refs. 8 and 9)

$$J = \sum_{k=1}^{N} \mathrm{Tr}\Big\{\big[Q_1 + MP_kH + (MP_kH)^T + (P_kH)^T Q_2 P_k H\big]X_k + (P_k V)^T Q_2 P_k V R\Big\} \tag{29}$$

Adjoin the covariance constraints Eq. (23) to the cost J using Lagrange multipliers, Λ_k, to obtain

$$\bar{J} = \sum_{k=1}^{N} \mathrm{Tr}\Big\{\big[Q_1 + MP_kH + (MP_kH)^T + (P_kH)^T Q_2 P_k H\big]X_k + (P_k V)^T Q_2 P_k V R + \Lambda_{k+1}^T\big[F_{ck}X_kF_{ck}^T + G_{ck}RG_{ck}^T - X_{k+1}\big]\Big\} \tag{30}$$

with $X_1 = X_{N+1}$.

Necessary conditions for minimum J are

$$\frac{\partial \bar{J}}{\partial X_k} = 0, \qquad \frac{\partial \bar{J}}{\partial \Lambda_{k+1}} = 0, \qquad \frac{\partial \bar{J}}{\partial \tilde{P}} = 0 \tag{31}$$

In addition, $\partial^2 \bar{J}/\partial \tilde{P}^2$ must be positive definite for a minimum J.

Substituting Eq. (30) into Eq. (31) and replacing P_k with $P_k = S_{1k}\tilde{P}S_{2k} + S_{3k}$ from Eq. (6), we obtain

$$\frac{\partial \bar{J}}{\partial X_k} = 0 = Q_1 + MP_kH + (MP_kH)^T + (P_kH)^T Q_2 P_k H + F_{ck}^T \Lambda_{k+1} F_{ck} - \Lambda_k \tag{32}$$

for $k = 1,2,\ldots,N$ with $\Lambda_k = \Lambda_{k+N}$.

$$\frac{\partial \bar{J}}{\partial \Lambda_{k+1}} = 0 = F_{ck}X_kF_{ck}^T + G_{ck}RG_{ck}^T - X_{k+1} \tag{33}$$

for $k = 1,2,\ldots,N$ with $X_k = X_{k+N}$.

$$\frac{\partial \bar{J}}{\partial \tilde{P}} = 0 = 2\sum_{k=1}^{N} S_{1k}^T \Big\{\big[Q_2 + G^T\Lambda_{k+1}G\big]P_k\big[HX_kH^T + VRV^T\big] + \big[M^T + G^T\Lambda_{k+1}F\big]X_kH^T\Big\}S_{2k}^T \tag{34}$$

Equations (32–34) are a set of coupled matrix equations. They make up necessary conditions for $\tilde{P}$, which is comprised of the multirate compensator gain matrices $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$, in Eqs. (1) and (2), to minimize the cost function J. Values of $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$, found by solving Eqs. (32–34), can be substituted into Eqs. (1) and (2), along with the definition of the sampling schedule, $s_{z,k}$, $s_{u,k}$, and $s_{y,k}$, to form the complete time-varying multirate compensator.

To ensure that the compensator gains satisfying Eqs. (32–34) minimize J, we should also check that the Hessian of $\bar{J}$ with respect to $\tilde{P}$ is positive definite. Our present algorithm does not calculate the Hessian explicitly, but uses an approximate value calculated by the numerical search algorithm discussed in the next section.

Equations (32–34) were derived assuming time-invariant compensator gains. We can easily derive the corresponding equations for periodically time-varying gains. Let

$$\bar{A} = \bar{A}_k, \qquad \bar{B} = \bar{B}_k, \qquad \bar{C} = \bar{C}_k, \qquad \bar{D} = \bar{D}_k \tag{35}$$

with the restriction that $\bar{A}_{k+N} = \bar{A}_k$, $\bar{B}_{k+N} = \bar{B}_k$, $\bar{C}_{k+N} = \bar{C}_k$, and $\bar{D}_{k+N} = \bar{D}_k$. Define the composite periodically time-varying compensator matrix

$$\tilde{P}_k = \begin{bmatrix} \bar{D}_k & \bar{C}_k \\ \bar{B}_k & \bar{A}_k \end{bmatrix} \tag{36}$$

Then replace $\tilde{P}$ with $\tilde{P}_k$ in Eq. (30) and differentiate with respect to $\tilde{P}_k$ to obtain

$$\frac{\partial \bar{J}}{\partial \tilde{P}_k} = 0 = S_{1k}^T \Big\{\big[Q_2 + G^T\Lambda_{k+1}G\big]P_k\big[HX_kH^T + VRV^T\big] + \big[M^T + G^T\Lambda_{k+1}F\big]X_kH^T\Big\}S_{2k}^T \quad \text{for } k = 1,2,\ldots,N \tag{37}$$

Thus, for every new set of compensator gains we obtain one new equation of the form of Eq. (37).

Equations (32–34) are very similar to the single-rate equations. In fact, if we set S_{1k}, S_{2k}, and S_{3k} so that they correspond to a single-rate system, and $N = 1$, we obtain the exact results derived by Mukhopadhyay for the single-rate case.[8]

Implementation

To find a reduced-order multirate compensator that minimizes the cost function J, we need to solve Eqs. (32–34) for the compensator gains $\tilde{P}$. A flowchart of the algorithm used to determine the compensator gains is shown in Fig. 3. Using the prescribed sampling schedule the algorithm first discretizes the analog plant model, analog cost function, and analog process noise model. (See Ref. 2 for a discussion of the relevant discretization procedures.) Equations (32–34) are then solved for the compensator gains using a gradient-type search in Mat-

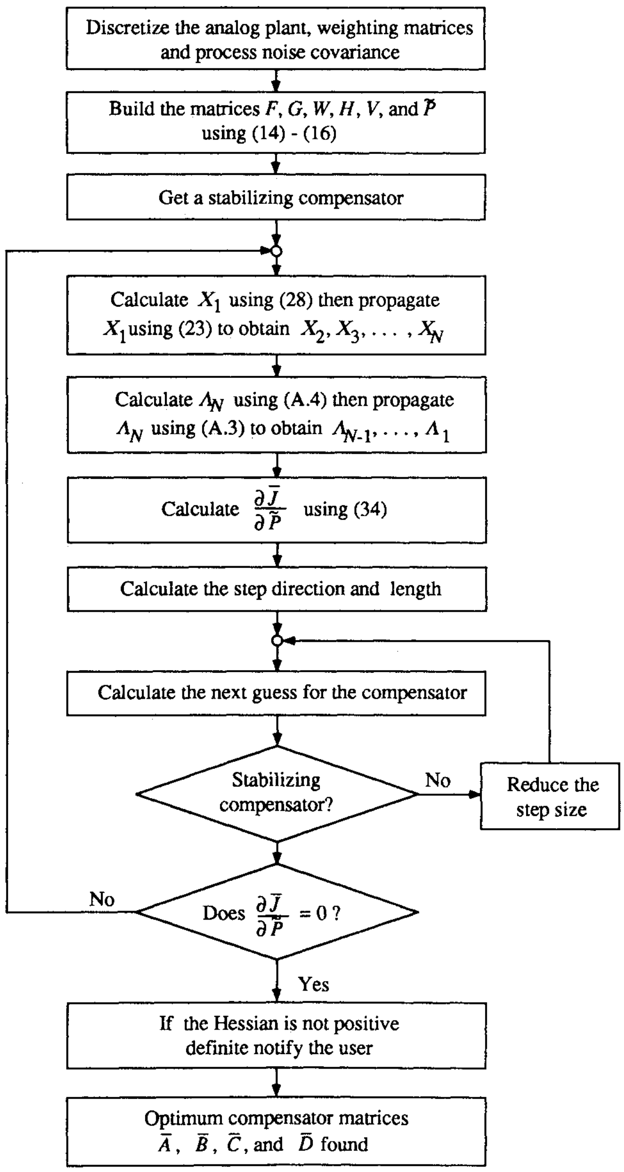
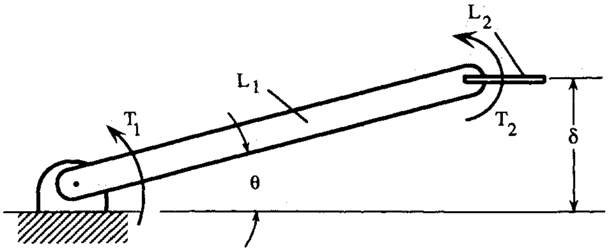


Fig. 3 Flowchart of optimization algorithm.



Parameters:

	Mass	Length
L_1	1.235 kg	0.965 m
L_2	0.163 kg	0.167 m

Inputs: Torque T_1 and T_2

Outputs: θ and δ

Fig. 4 Planar two-link robot arm.

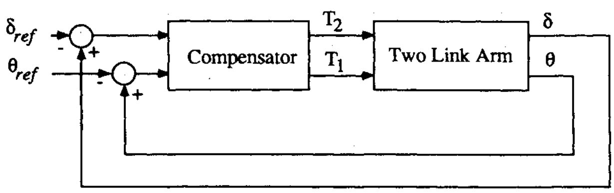


Fig. 5 TLA plant/compensator configuration.

lab.[12] We chose a gradient-type search to solve Eqs. (32–34) because it allows us to easily add constraints on the parameters values—simple equality constraints were used to find the optimized compensators in the next section. The equations necessary to solve for the Lagrange multipliers are located in the Appendix, Eqs. (A3) and (A4). To ensure that the solution represents a minimum J, the algorithm checks that the Hessian of J with respect to the free parameters in $\tilde{P}$ is positive definite at the solution point.

Because Eqs. (32–34) are not valid when the closed-loop system is unstable, the algorithm 1) must be provided with an initial stabilizing compensator, and 2) must result in a stabilizing compensator at every iteration. From our experience, finding an initial stabilizing compensator is generally not a problem. Many systems suitable for multirate control can be stabilized using successive loop closure with minimal cross coupling between the control loops. A stabilizing multirate compensator can then be obtained by discretizing the individual continuous control loops at the desired sampling rates. When there are no constraints on its structure, a stabilizing compensator can also be obtained using the boot strapping method of Boussard.[13,14] For difficult multirate control problems, where a stabilizing compensator cannot be found using either of the preceeding two methods, one can always use Yang's algorithm to find a stabilizing compensator and then switch to our algorithm to complete the optimization. In our experience, Yang's algorithm usually converges to a stabilizing solution quickly—it is the optimization of the compensator parameters that is time consuming.

To avoid the problem of destabilizing compensators during the iteration process, we included a check in the algorithm which systematically reduces the step size to ensure that the compensator is stabilizing. Because the gradient of the cost function with respect to the compensator parameters becomes very large near the stability boundary, the algorithm is always forced away from a destabilizing solution as long as it never steps over the stability boundary into an unstable region.

Even though our algorithm was programmed as an interpreted Matlab M-File we found that it still performed better than Yang's algorithm which runs as compiled Fortran. The primary difference between the two algorithms is in the complexity of the expression for the gradient of $\bar{J}$ with respect to the compensator parameters. Calculation of the gradient expression for Yang's problem involves diagonalization of the closed-loop system and evaluation of several matrix equations with nested summations. Compare Eqs. (32–34) with Eqs. (112–115) in Ref. 3 to see the difference in the complexity of the two gradient expressions.

Table 1 Sampling/update rates for TLA

T_2	Sample/update rate, s
θ	0.225
δ	0.028125
T_1	0.225
T_2	0.028125

Two-Link Robot Arm Example

We used a mathematical model of a planar two-link robot arm (TLA) to demonstrate the capabilities of our algorithm. This is the same model used by Yang,[6] and so we were able to verify our results by direct comparison. A diagram of the TLA is shown in Fig. 4.

The goal of our design was to control the tip position δ of the arm via a multirate compensator. We used the following analog cost function and process noise covariance matrices from Ref. 6.

$$J = \lim_{t\to\infty} E\left\{ x^T \begin{bmatrix} 0.21 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 18.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + u^T \begin{bmatrix} 0.01 & 0 \\ 0 & 0.69444 \end{bmatrix} u \right\} \tag{38}$$

where

$$x = \begin{Bmatrix} \theta \\ \dot{\theta} \\ \delta \\ \dot{\delta} \end{Bmatrix}, \qquad u = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$E\{ww^T\} = \begin{bmatrix} 0.69444 & 0 \\ 0 & 0.01 \end{bmatrix} \tag{39}$$

We assumed perfect measurement and that plant disturbances enter the system coincident with the control torques. The sampling/update rates are given in Table 1.

Five difference compensators were designed: an analog LQR, a multirate lead/lead, an optimized multirate lead/lead, an optimized multirate general second order, and an optimized single-rate general second order. We used a smooth step input to δ_{ref} and θ_{ref} defined as follows:

$$\delta_{\text{ref}}(t) = \begin{cases} 0.005\left[1 - \cos\left(\dfrac{\pi t}{T_c}\right)\right] \text{m}, & t \le T_c \\ 0.001 \text{ m}, & t \ge T_c \end{cases}$$

$$\theta_{\text{ref}}(t) = \frac{\delta_{\text{ref}}(t)}{L_1 + L_2}, \qquad T_c = 0.125 \text{ s} \tag{40}$$

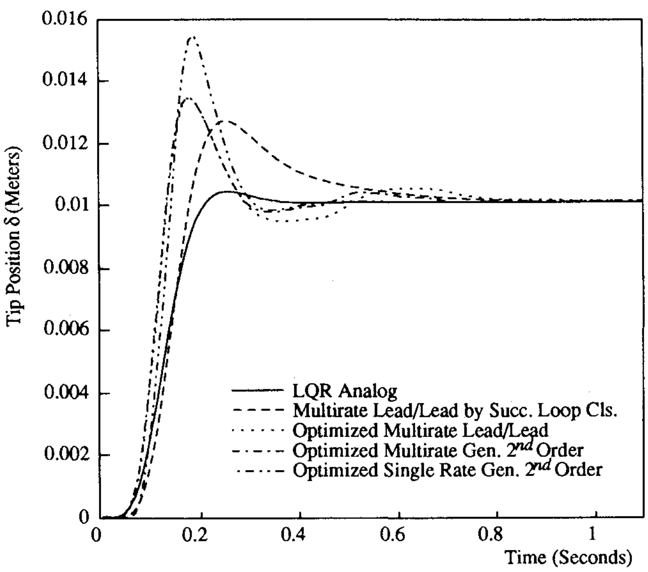


Fig. 6a Tip δ response to a smooth step command to tip position.

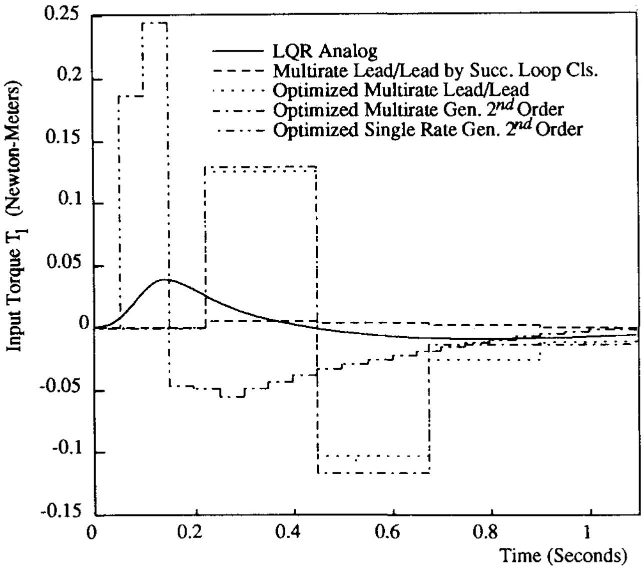


Fig. 6b Control torque T_1 response to a smooth step command to tip position.

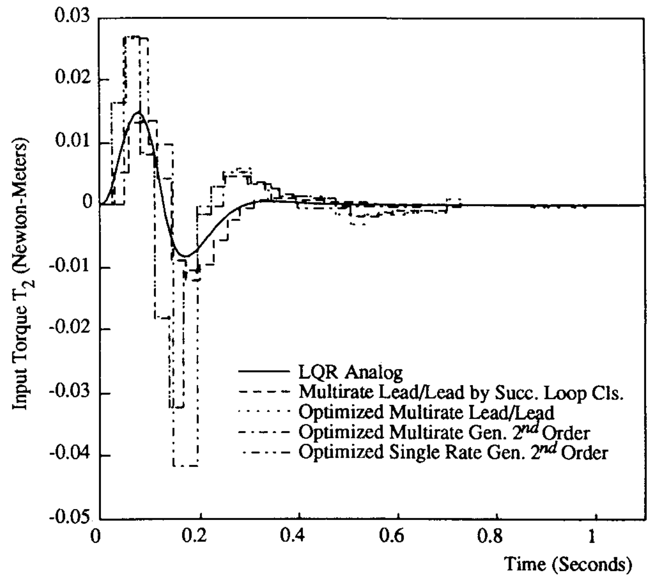


Fig. 6c Control torque T_2 response to a smooth step command to tip position.

and the servo configuration shown in Fig. 5 to measure the performance of the different compensators. The response of the TLA for the five compensators is shown in Figs. 6a–6c.

The analog LQR compensator used full state feedback. We provided this compensator as an example of the response possible using the cost function weighting matrices of Eq. (38).

The multirate lead/lead was found using successive loop closures. We designed the control loops in the discrete domain so that the eigenvalues of the closed-loop system matched those we obtained using LQR transformed to discrete time. This compensator consists of two simple lead loops: one from δ to T_2 operating at the fast sampling/update rate, and one from θ to T_1 operating at the slow sampling/update rate.

The final three compensators were synthesized using our new algorithm and the cost-weighting matrices used to design the analog LQR compensator. The optimized multirate lead/lead was found by optimizing the pole/zero locations and gains of the lead/lead compensator found by successive loop closures.

The optimized multirate general second-order compensator uses the same sampling/update scheme as the lead/lead compensators but has the compensator structure of Eq. (41), where a_{ij}, b_{ij}, c_{ij}, and d_{ij} are the parameters which were optimized. This compensator has the maximum number of independent free parameters possible for a second-order system.[7]

$$\bar{A}=\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22}\end{bmatrix},\quad \bar{B}=\begin{bmatrix} 1 & b_{12} \\ b_{21} & 1\end{bmatrix}$$

$$\bar{C}=\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{bmatrix},\quad \bar{D}=\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22}\end{bmatrix} \tag{41}$$

The optimized single-rate general second-order compensator is a single-rate equivalent of the multirate general second-order compensator. It has the same structure as the multirate general second-order compensator, Eq. (41), but uses a single sampling rate. This sampling rate was chosen such that the number of computations required to implement either the multirate or single-rate compensators during real-time operation are the same.

Our results are the same as those obtained using Yang's algorithm. They demonstrate how multirate compensators can provide better performance than single-rate compensators by trading lower bandwidth control of the slow modes for higher bandwidth control of the fast modes. In this example, we were able to reduce the tip response overshoot 40% and the peak control torque 25% by using a multirate controller over a single-rate controller.

Conclusions

In this paper, we have presented a new algorithm for synthesizing reduced-order multirate compensators. It can be used to design compensators of arbitrary structure and dynamic order, with independent sampling/update rates for the compensator inputs, outputs, and states. This algorithm provides the versatility of Yang's algorithm without the numerical complexities associated with the finite time cost function.

Finally, we do not want to discount Yang's algorithm altogether because, while our algorithm requires an initial stabilizing compensator, Yang's does not. For those problems where finding an initial stabilizing compensator is difficult, we can always use Yang's algorithm to find a stabilizing compensator and then quickly optimize the compensator parameter values with our algorithm.

Appendix

Given a P_k which stabilizes the multirate system, we can calculate the steady-state values of Λ_k where Λ_k is defined by Eq. (32) rewritten here as Eq. (A1).

$$0 = Q_1 + MP_kH + (MP_kH)^T + (P_kH)^TQ_2P_kH + F_{ck}^T\Lambda_{k+1}F_{ck} - \Lambda_k \tag{A1}$$

for $k = 1,2,\ldots,N$ with $\Lambda_k = \Lambda_{k+N}$.

First simplify Eq. (A1) by defining

$$Q_3 \equiv \begin{bmatrix} Q_1 & M \\ M^T & Q_2 \end{bmatrix}, \qquad J_k \equiv \begin{bmatrix} I \\ P_k H \end{bmatrix} \tag{A2}$$

where I is an identity matrix. Then Eq. (A1) can be written as

$$\Lambda_k = J_k^T Q_3 J_k + F_{ck}^T \Lambda_{k+1} F_{ck} \tag{A3}$$

for $k = 1,2,\ldots,N$ with $\Lambda_k = \Lambda_{k+N}$.

Equation (A3) represents a periodically time-varying Lyapunov equation. We can create an equivalent single-rate system by repeated application of Eq. (A3).

$$\Lambda_k = J_{dk}^T Q J_{dk} + F_{dk}^T \Lambda_k F_{dk} \tag{A4}$$

for $k = 1,2,\ldots,N$ with $\Lambda_k = \Lambda_{k+N}$.

$$F_{dk} = F_{c(k+N-1)} F_{c(k+N-2)} F_{c(k+N-3)} \cdots F_{ck} \tag{A5}$$

$$J_{dk} = \begin{bmatrix} J_{(k+N-1)} F_{c(k+N-2)} F_{c(k+N-3)} \cdots F_{ck} \\ J_{(k+N-2)} F_{c(k+N-3)} \cdots F_{ck} \\ \vdots \\ J_k \end{bmatrix} \tag{A6}$$

$$Q_d = \begin{bmatrix} Q_3 & 0 & \cdots & 0 \\ 0 & Q_3 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & Q_3 \end{bmatrix}$$

Equation (A4) is a time-invariant Lyapunov equation which can be solved for Λ_k. Once any Λ_k has been found, the propagation Eq. (A3) can be used to find the remaining Λ_k.

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